APPLICATIONS OF OPERATOR SPACE THEORY TO NEST ALGEBRA BIMODULES

G. K. ELEFTHERAKIS

Abstract. Recently, Blecher and Kashyap have generalized the notion of W^*-modules over von Neumann algebras to the setting where the operator algebras are $\sigma-$weakly closed algebras of operators on a Hilbert space. They call these modules *weak* rigged modules*. We characterize the weak* rigged modules over nest algebras. We prove that Y is a right weak* rigged module over a nest algebra $\mathrm{Alg}(\mathcal{M})$ if and only if there exists a completely isometric normal representation Φ of Y and a nest algebra $\mathrm{Alg}(\mathcal{N})$ such that $\mathrm{Alg}(\mathcal{N})\Phi(Y)\mathrm{Alg}(\mathcal{M}) \subset \Phi(Y)$ while $\Phi(Y)$ is implemented by a continuous nest homomorphism from $\mathcal{M}$ onto $\mathcal{N}$. We describe some properties which are preserved by continuous CSL homomorphisms.

1. Introduction

C^*-modules are well known as a generalization of the notion of Hilbert spaces introduced by Kaplansky [12]. They are very important tools for the study of C^*-algebras. The dual version is the notion of W^*-module. A W^*-module over a von Neumann algebra is a C^*-module which is a dual Banach space and defines a separately weak* continuous inner product [4]. W^*-modules were originally introduced by Paschke [14]. They possess fruitful properties and appear in many versions such as weak* ternary rings of operators (TROs).

The development of operator space theory makes the generalization of W^*-modules possible even in the case of non-selfadjoint dual operator algebras. In particular, Blecher and Kashyap provided a new characterization of W^*-modules which enabled the definition of an analogous notion for modules over non-selfadjoint dual operator algebras [1]. They call these modules *weak* rigged modules*.

In the present paper we characterize the weak* rigged modules over nest algebras. We prove that a right dual operator module Y over a nest algebra B is a nest algebra bimodule, i.e., there exists a nest algebra A such that $AYB \subset Y$. Nest algebra bimodules are well constructed and described in [11]: If $\mathrm{Alg}(\mathcal{N}), \mathrm{Alg}(\mathcal{M})$ are nest algebras and Y is a weak* closed space satisfying $\mathrm{Alg}(\mathcal{N})Y\mathrm{Alg}(\mathcal{M}) \subset Y$, there exists an order preserving left continuous map

$\theta : \mathcal{M} \to \mathcal{N}$ such that $Y = Op(\theta)$, where

$$Op(\theta) = \{y : \theta(M)^{\perp} y M = 0 \ \ \forall M \ \in \ \mathcal{M}\}. \tag{1.1}$$

We prove that Y is a right weak* rigged module over $\mathrm{Alg}(\mathcal{M})$ if and only if there exists a normal completely isometric representation Φ and a nest algebra $\mathrm{Alg}(\mathcal{N})$ such that $\Phi(Y) = Op(\theta)$ where $\theta : \mathcal{M} \to \mathcal{N}$ is a left-right continuous, therefore continuous, surjective nest homomorphism.

We outline the results of this paper: In Section 2 we fix a continuous nest surjective homomorphism $\theta : \mathcal{M} \to \mathcal{N}$. We prove that $Y = Op(\theta)$, (see (1.1)) is a right weak* rigged module over $\mathrm{Alg}(\mathcal{M})$. There also exists an $\mathrm{Alg}(\mathcal{N}) - \mathrm{Alg}(\mathcal{M})$ module X such that $\mathrm{Alg}(\mathcal{N})$ is isomorphic as an $\mathrm{Alg}(\mathcal{N})$-module to the space $Y \otimes^{\sigma h}_{\mathrm{Alg}(\mathcal{M})} X$ which is the $\mathrm{Alg}(\mathcal{M})$ balanced normal Haagerup tensor product of Y and X, [10]. Another result is that the algebra $\mathrm{Alg}(\mathcal{N})$ is isomorphic to the algebra of the left multipliers over $Y, M_l(Y)$.

In Section 3 we prove the converse: If Y is an abstract right weak* rigged module over the nest algebra $\mathrm{Alg}(\mathcal{M})$, there exists a normal completely isometric representation Φ of Y, a nest algebra $\mathrm{Alg}(\mathcal{N})$, and a continuous surjective nest homomorphism $\theta : \mathcal{M} \to \mathcal{N}$ with $\Phi(Y) = Op(\theta)$.

In Section 4 we present some examples and counterexamples of right weak* rigged modules over nest algebras. In the counterexamples we will see that the continuity of the nest is important.

In Section 5, inspired by the nest algebra case, we define the notion of a spatial embedding of a dual operator algebra A in another dual operator algebra B. In case A and B are CSL algebras which correspond to CSLs $\mathcal{L}_1, \mathcal{L}_2$, we prove that A is spatially embedded in B if and only if there exists a continuous CSL homomorphism from $\mathcal{L}_2$ onto $\mathcal{L}_1$. Three natural consequences are the following:

(i) Let $\mathcal{K}(\mathrm{Alg}(\mathcal{L}_2))$, (resp. $\mathcal{K}(\mathrm{Alg}(\mathcal{L}_1))$) be the subalgebra of $\mathrm{Alg}(\mathcal{L}_2)$ (resp. $\mathrm{Alg}(\mathcal{L}_1)$) which contains its compact operators. Then

$$\mathrm{Alg}(\mathcal{L}_2) = \overline{\mathcal{K}(\mathrm{Alg}(\mathcal{L}_2))}^{w^*} \Rightarrow \mathrm{Alg}(\mathcal{L}_1) = \overline{\mathcal{K}(\mathrm{Alg}(\mathcal{L}_1))}^{w^*}.$$

(ii) If $\mathrm{Alg}(\mathcal{L}_1)$ contains a non-zero compact operator (resp. finite rank operator), then $\mathrm{Alg}(\mathcal{L}_2)$ also contains a non-zero compact operator (resp. finite rank operator).

(iii) If $\mathcal{L}_2$ is a synthetic lattice, then $\mathcal{L}_1$ is also a synthetic lattice.

In the following paragraphs we describe the notions we use in this paper; since we use extensively the basics of operator space theory, we refer the reader to the monographs [4], [6], [15], and [16] for details.

Let H and K be Hilbert spaces and $A \subset B(H)$ be an algebra. A subspace $X \subset B(K, H)$ is called a left module over A if $AX \subset X$. Similarly we can define right module over A. A left and right module over A is called a bimodule

over A. An operator Y is an abstract left (right) operator module over an abstract operator algebra A if there exists a completely contractive bilinear map $A \times Y \to Y$ $(Y \times A \to Y)$. A left and right operator module over A is called an operator bimodule over A.

Let A be a dual operator algebra and Y be a dual operator space. We say that Y is a left (right) dual operator module if the above completely contractive bilinear map is separately weak* continuous. A left and right dual operator module over A is called a dual operator bimodule over A.

Two operator bimodules Y and Z over an operator algebra A are called isomorphic as operator bimodules if there exists a completely isometric and surjective $A-$bimodule map $\pi : Y \to Z$. This is then written $Y \cong Z$. If A is a dual operator algebra and Y and Z are dual operator $A-$bimodules, we write $Y \cong Z$ if also π is weak* (bi)continuous.

If Y is a dual right operator module over a dual operator algebra B and X is a left dual operator module over B, we denote by $Y \otimes_B^{\sigma h} X$ the balanced normal Haagerup tensor product of Y and X which linearizes the separately weak* continuous completely bounded $B-$balanced bilinear maps [10]. If Y (resp. X) is a left (resp. right) dual operator module over a dual operator algebra A, then $Y \otimes_B^{\sigma h} X$ is also a left (resp. right) dual operator module over A, [10].

The following is the definition of Morita equivalence used in this paper:

Definition 1.1. [1] *The dual operator algebras B, A are called* ***weakly* Morita equivalent*** *if there exist a $B-A$ dual operator module X and an $A-B$ dual operator module Y such that $B \cong X \otimes_A^{\sigma h} Y$ and $A \cong Y \otimes_B^{\sigma h} X$ as B and A dual operator bimodules respectively.*

Fix $n, m \in \mathbb{N}$. If X is an operator space, $M_{m,n}(X)$ is the operator space of $m \times n$ matrices with entries in X. Then $M_n(X) = M_{n,n}(X)$, $C_n(X) = M_{n,1}(X)$, $R_n(X) = M_{1,n}(X)$.

If X is a subspace of $B(H, K)$, where H and K are Hilbert spaces, we denote by $R_\infty^{fin}(X)$ (resp. $C_\infty^{fin}(X)$) the space of operators $(x_1, x_2, ...) : H^\infty \to K$ (resp. $(x_1, x_2, ...)^T : H \to K^\infty$) such that $x_i \in X$ for all i and there exists $n_0 \in \mathbb{N}$ for which $x_n = 0$ for all $n \geq n_0$.

In this paper we shall use the notion of the multiplier algebra $M_l(Y)$ of a dual operator space Y [4, 4.5.1]. We recall that $M_l(Y)$ is a unital dual operator algebra and Y is a left dual operator module over $M_l(Y)$. Every $u \in M_l(Y)$ is a weak* continuous map from Y into Y. A bounded net $(u_t)_t \subset M_l(Y)$ converges in the weak* topology to $u \in M_l(Y)$ iff $u_t(y) \stackrel{w^*}{\to} u(y)$ for all $y \in Y$.

We present the definition of right weak* rigged modules over dual operator algebras. Throughout this paper we shall use this definition or other equivalent definitions from [2].

Definition 1.2. *Suppose that Y is a dual operator space and a right module over a dual operator algebra B. Suppose that there exists a net of positive integers $(n(\alpha))$ and weak* continuous completely contractive $B-$module maps*

$$\phi_\alpha : Y \longrightarrow C_{n(\alpha)}(B), \quad \psi_\alpha : C_{n(\alpha)}(B) \to Y,$$

with $\psi_\alpha(\phi_\alpha(y)) \to y$ in the weak topology on Y, for all $y \in Y$. Then we say that Y is a* ***right weak* rigged module over*** *B.*

If $\mathcal{L}$ is a set of projections of a Hilbert space H we denote by $\mathrm{Alg}(\mathcal{L})$ the algebra

$$\{x \in B(H) : N^\perp x N = 0 \ \forall \ N \ \in \mathcal{L}\}.$$

A **nest** $\mathcal{N}$ is a totally ordered set of projections of a Hilbert space H containing the zero and identity operators which is closed under arbitrary intersections and closed spans. The corresponding **nest algebra** is $\mathrm{Alg}(\mathcal{N})$. If $N \in \mathcal{N}$, we denote by N_- the projection onto the closed span of the union $\cup_{\substack{M<N \\ M \in \mathcal{N}}}(M(H))$. If $N_- < N$ we call the projection $N \ominus N_-$ an **atom**. If a nest has no atoms, it is called a **continuous nest**. If the atoms span the identity operator, the nest is called a **totally atomic nest**. An order preserving map between two nests is called a **nest homomorphism**. If this map is injective and surjective, it is called a **nest isomorphism**.

If $\mathcal{N}_1$ and $\mathcal{N}_2$ are nests acting on the Hilbert spaces H_1, H_2 respectively, and if $\theta : \mathcal{N}_1 \to \mathcal{N}_2$ is a nest homomorphism, we denote by $Op(\theta)$ the space of operators $x \in B(H_1, H_2)$ satisfying $\theta(N)^\perp x N = 0$ for all $N \in \mathcal{N}_1$. Observe that $Op(\theta)$ is an $\mathrm{Alg}(\mathcal{N}_2)-\mathrm{Alg}(\mathcal{N}_1)$ bimodule. If Y is a subspace of $B(H_1, H_2)$, the map sending every projection p of H_1 to the projection of H_2 generated by the vectors of the form $yp(\xi) : y \in Y, \xi \in H_1$ is denoted by $\mathrm{Map}(Y)$.

Finally, if X is a normed space, we denote by $Ball(X)$ the unit ball of X.

We recall the following results which will be used in this paper:

Theorem 1.1. [11] *Let $\mathcal{N}_1, \mathcal{N}_2$ be nests, Y be a weak* closed $\mathrm{Alg}(\mathcal{N}_2) - \mathrm{Alg}(\mathcal{N}_1)$ bimodule, and θ be the restriction of $\mathrm{Map}(Y)$ to $\mathcal{N}_1$. Then θ is left continuous, $\theta(\mathcal{N}_1) \subset \mathcal{N}_2$, and $Y = Op(\theta)$.*

Theorem 1.2. [9] *Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be nests, B and A be the corresponding nest algebras, $\theta : \mathcal{N}_1 \to \mathcal{N}_2$ be a nest isomorphism, and $Y = Op(\theta), X = Op(\theta^{-1})$. Then*

(i) $B = \overline{\mathrm{span}}^{w^*}(XY)$ and $A = \overline{\mathrm{span}}^{w^*}(YX)$.

(ii) B and A are weakly Morita equivalent. In particular, $B \cong X \otimes^{\sigma h}_A Y$ as dual operator $B-$modules, and $A \cong Y \otimes^{\sigma h}_B X$ as dual operator $A-$modules.*

The last theorem implies, [1, Theorem 3.3], the following:

Theorem 1.3. *Let A, B, X, Y be as in Theorem 1.2. There exist nets $(y_t)_t \subset Ball(R^{fin}_\infty(Y))$,$(x_t)_t \subset Ball(C^{fin}_\infty(X))$ such that $y_t x_t \stackrel{w^*}{\to} I_A$ where I_A is the*

identity of A, *and nets* $(u_i)_i \subset Ball(R^{fin}_\infty(X))$,$(w_i)_i \subset Ball(C^{fin}_\infty(Y))$ *such that* $u_i w_i \stackrel{w^*}{\to} I_B$ *where* I_B *is the identity of* B.

2. Continuous nest homomorphisms

Let $\mathcal{N}, \mathcal{M}$ be nests acting on the Hilbert spaces L, H respectively, A and B be the corresponding nest algebras, and $\phi : \mathcal{M} \to \mathcal{N}$ be a nest homomorphism. We assume that $\mathcal{N}$ is generated as a nest by $\phi(\mathcal{M})$. Observe that if ϕ is continuous, then $\phi(\mathcal{M}) = \mathcal{N}$. We define the following spaces:

$$Y = Op(\phi) = \{y \in B(H, L) : \phi(M)^\perp y M = 0, \ \ \forall \ \ M \in \mathcal{M}\}$$

$$X = \{x \in B(L, H) : M^\perp x \phi(M) = 0, \ \ \forall \ \ M \in \mathcal{M}\}.$$

Observe that Y is an $A - B$ bimodule and X is a $B - A$ bimodule.

Theorem 2.1. *If* ϕ *is continuous, then*
(i) $A = \overline{\text{span}}^{w^*}(YX)$,
(ii) the space Y *is a right weak* rigged module over* B,
(iii) the algebra A *is isomorphic as a dual* $A-$*operator module with* $Y \otimes^{\sigma h}_B X$,
(iv) the algebra A *is isomorphic as a dual operator algebra with the algebra* $M_l(Y)$ *of left multipliers over* Y.

Proof

We define the nests

$$\mathcal{L}_1 = \mathcal{M}^{(2)}, \quad \mathcal{L}_2 = \{M \oplus \phi(M) : M \in \mathcal{M}\}$$

and the corresponding nest algebras

$$\mathrm{Alg}(\mathcal{L}_1) = \begin{pmatrix} B & B \\ B & B \end{pmatrix}$$

$$\mathrm{Alg}(\mathcal{L}_2) = \begin{pmatrix} B & X \\ Y & A \end{pmatrix}.$$

The map $\psi : \mathcal{L}_1 \to \mathcal{L}_2 : M \oplus M \longrightarrow M \oplus \phi(M)$ is a nest isomorphism. We can easily check that

$$\hat{Y} = Op(\psi) = \begin{pmatrix} B & B \\ Y & Y \end{pmatrix}$$

and

$$\hat{X} = Op(\psi^{-1}) = \begin{pmatrix} B & X \\ B & X \end{pmatrix}.$$

Theorem 1.2 implies that $\mathrm{Alg}(\mathcal{L}_2) = \overline{\text{span}}^{\mathrm{w}^*}(\hat{Y}\hat{X})$. Therefore $A = \overline{\text{span}}^{\mathrm{w}^*}(YX)$. Using Theorem 1.3 we find nets

$$(\hat{y}_t)_t \subset Ball(R^{fin}_\infty(\hat{Y})), \quad (\hat{x}_t)_t \subset Ball(C^{fin}_\infty(\hat{X}))$$

with the property $\hat{y}_t \hat{x}_t \stackrel{w^*}{\to} I_{\mathrm{Alg}(\mathcal{L}_2)}$.

We may take

$$\hat{y}_t = \begin{pmatrix} b_t^1 & b_t^2 \\ y_t^1 & y_t^2 \end{pmatrix}$$

where $b_t^1, b_t^2 \in R_\infty^{fin}(B), \quad y_t^1, y_t^2 \in R_\infty^{fin}(Y)$ and

$$\hat{x}_t = \begin{pmatrix} c_t^1 & x_t^1 \\ c_t^2 & x_t^2 \end{pmatrix}$$

where $c_t^1, c_t^2 \in C_\infty^{fin}(B), \quad x_t^1, x_t^2 \in C_\infty^{fin}(X)$.

Put

$$y_t = (y_t^1, \quad y_t^2) \in R_\infty^{fin}(Y), \quad x_t = (x_t^1, \quad x_t^2)^T \in C_\infty^{fin}(X).$$

Since $\hat{y}_t$ and $\hat{x}_t$ are contractions, y_t and x_t are also contractions. Clearly the net

$$y_t x_t = y_t^1 x_t^1 + y_t^2 x_t^2$$

converges to the identity of the algebra A. We deduce that Y is a right weak* rigged module over B. (See the fourth description of the weak* rigged modules in [2]).

Statements (iii) and (iv) follow from (ii) using the results of [2]. We include a proof for completeness.

The map

$$Y \times X \to A, \quad (y, x) \to yx$$

is a complete contraction, weak* continuous, and an $A-$module map. So it induces a map

$$\rho : Y \otimes_B^{\sigma h} X \to A$$

which is also a complete contraction, weak* continuous, and an $A-$module map. We claim that ρ is $1-1$. Indeed, if $y \in Y$ and $x \in X$, then

$$(y \otimes_B x) y_t x_t = y \otimes_B (x y_t x_t) = (y x y_t) \otimes_B x_t = \rho(y \otimes_B x)(y_t \otimes_B x_t).$$

It follows that

$$w y_t x_t = \rho(w)(y_t \otimes_B x_t) \quad \forall w \in Y \otimes_B^{\sigma h} X.$$

Since $(y_t x_t)_t$ converges to the identity operator, $\rho(w) = 0$ implies $w = 0$.

If $v_1, v_2, ..., v_m \in X, w_1, w_2, ..., w_m \in Y$, we have

$$\left\| \sum_{k=1}^m w_k \otimes_B (v_k y_t x_t) \right\| = \left\| (\sum_{k=1}^m w_k v_k y_t) \otimes_B x_t \right\|$$

$$\leq \left\| \sum_{k=1}^m w_k v_k y_t \right\| \|x_t\| \leq \left\| \sum_{k=1}^m w_k v_k \right\|$$

Letting $y_t x_t \to I_A$, we have

$$\left\| \sum_{k=1}^{m} w_k \otimes_B v_k \right\| \leq \left\| \sum_{k=1}^{m} w_k v_k \right\|.$$

We proved that the restriction of ρ to

$$Y \otimes_B^h X = \overline{span}^{\|\cdot\|}\{y \otimes_B x : y \in Y, x \in X\}$$

is an isometry. Choose $u \in Y \otimes_B^{\sigma h} X$ and a net $(u_k)_k$ in $Y \otimes_B^h X$ converging to u in the weak* topology. For any finite rank operators $g, f \in Ball(A)$ we have

$$\rho(f u_k g) = f \rho(u_k) g \xrightarrow{\|\cdot\|} f \rho(u) g = \rho(f u g).$$

Since ρ is $1-1$, we conclude that $fug \in Y \otimes_B^h X$, and so

$$\|fug\| = \|\rho(fug)\| = \|f\rho(u)g\| \leq \|\rho(u)\|.$$

The identity of A is in the weak* closure of its finite rank contractions, [5]. Therefore $\|u\| \leq \|\rho(u)\|$. We proved that ρ is an isometry. Similarly we can prove that it is a complete isometry.

It remains to prove statement (iv). Define $\sigma : A \to M_l(Y)$ by

$$\sigma(a)(y) = ay \ \ \forall \ \ a \in A, y \in Y$$

which is clearly contraction. Let u be in $M_l(Y)$. Since u is a weak* continuous $B-$module homomorphism,we have

$$\lim_t \sigma(u(y_t)x_t)(y) = \lim_t u(y_t)x_t y = \lim_t u(y_t x_t y) = u(y), \ \ \forall \ \ y \in Y.$$

It follows that $(\sigma(u(y_t)x_t))_t$ converges in the weak* topology of $M_l(Y)$ to u. So $u \in \overline{\sigma(A)}^{w^*}$. If $a \in A$,

$$\|a y_t x_t\| = \|\sigma(a)(y_t)x_t\| \leq \|\sigma(a)\|.$$

The equality $\|a\| = \|\sigma(a)\|$ follows from the fact $a y_t x_t \xrightarrow{w^*} a$. So σ is an isometry. Similarly we can prove that it is a complete isometry. The Krein–Smulian Theorem, [4, A.2.5], implies that

$$M_l(Y) = \overline{\sigma(A)}^{w^*} = \sigma(A). \qquad \square$$

Theorem 2.2. *The following are equivalent:*

(i) ϕ is continuous;

(ii) $A = \overline{\mathrm{span}}^{w^*}(YX)$;

(iii) The identity of the Hilbert space L belongs to $\overline{\mathrm{span}}^{w^*}(YX)$.

Proof Statement (i) implies (ii) by Theorem 2.1. Since nest algebras are unital, (ii) implies (iii). Suppose that $I_L \in \overline{\mathrm{span}}^{\mathrm{w}^*}(YX)$. There exist nets $(w_t)_t \subset R_\infty^{fin}(Y), (u_t)_t \subset C_\infty^{fin}(X)$ such that $w_t u_t \xrightarrow{w^*} I_A$. If $(M_i)_i$ is a decreasing net of projections in $\mathcal{M}$ converging in M, we may assume that the

net $(\phi(M_i))_i$ converges in $N \in \mathcal{N}$. Clearly $\phi(M) \leq N$. We shall prove that $N \leq \phi(M)$.

$$N = \lim_t w_t u_t N = \lim_t (\lim_i w_t u_t \phi(M_i)) = \lim_t (\lim_i w_t M_i^\infty u_t \phi(M_i)).$$

Since $(M_i^\infty)_i$ (resp. $(\phi(M_i))_i$) converges in the strong operator topology to M^∞ (resp. to N),

$$\begin{aligned} N = & \lim_t w_t M^\infty u_t N = \lim_t \phi(M) w_t M^\infty u_t N \\ = & \phi(M) \lim_t (w_t M^\infty u_t N) \Rightarrow N \leq \phi(M). \end{aligned}$$

We proved that $\phi(M) = N$. Therefore ϕ is right continuous. The proof of the fact ϕ is left continuous is similar. $\square$

Let $\widetilde{X}$ be the space ${}_{w^*}CB(Y, B)_B$. This is the space of weak* continuous completely bounded $B-$module maps from Y to B and it is, [1], a dual $B-A$ module under the actions

$$B \times \widetilde{X} \longrightarrow \widetilde{X} : (b, u) \to b \cdot u, \quad b \cdot u(y) = bu(y) \ \ \forall y \in Y,$$
$$\widetilde{X} \times A \longrightarrow \widetilde{X} : (u, a) \to u \cdot a, \quad u \cdot a(y) = u(ay) \ \ \forall y \in Y.$$

Theorem 2.3. *Suppose that ϕ is continuous. Then:*

(i) The map $\iota : X \to \widetilde{X}$: $\iota(x)(y) = xy$ is a weak continuous completely isometric surjective map.*

(ii) $M^\perp \cdot \widetilde{X} \cdot \phi(M) = 0$ or equivalently $M^\perp u(\phi(M)y) = 0$ for all $M \in \mathcal{M}$, $u \in \widetilde{X}$, $y \in Y$.

Proof Clearly ι is a complete contraction. By Theorem 2.1 there exist nets

$$(y_t)_t \subset Ball(R_\infty^{fin}(Y)), (x_t)_t \subset Ball(C_\infty^{fin}(X))$$

such that $y_t x_t \stackrel{w^*}{\to} I_A$. So for all $x \in X$ we have

$$\|x y_t x_t\| = \|\iota(x)(y_t) x_t\| \leq \|\iota(x)\|.$$

Since $w^* - \lim_t x y_t x_t = x$, we have $\|x\| \leq \|\iota(x)\|$. Hence ι is an isometry. Similarly, we can prove that ι is a complete isometry. If $u \in \widetilde{X}$ for all $y \in Y$,

$$\begin{aligned} & u(y_t x_t y) = u(y_t) x_t y = u(y_t) \iota(x_t)(y) \\ \Rightarrow & u(y) = \lim_t u(y_t) \iota(x_t)(y) = \lim_t \iota(u(y_t) x_t)(y). \end{aligned}$$

Thus

$$u = w^* - \lim_t \iota(u(y_t x_t)) \in \overline{\iota(X)}^{w^*}$$

in the weak* topology of $CB(Y, B)$. By the Krein–Smulian Theorem $\iota(X) = \widetilde{X}$, so statement (i) holds. Since $M^{\perp}X\phi(M) = 0$ for all $M \in \mathcal{M}$, (i) implies (ii). $\square$

Theorem 2.5 describes how we can construct right weak* rigged modules over nest algebras. We first need the following lemma:

Lemma 2.4. *Let $\mathcal{M}, \mathcal{N}$ be nests acting on Hilbert spaces H, L respectively, and let $B = \mathrm{Alg}(\mathcal{M}), A = \mathrm{Alg}(\mathcal{N})$ be the corresponding nest algebras. Suppose that there exist spaces X_0, Y_0 such that:*
(i) $X_0AY_0 \subset B$;
(ii) The identity of the Hilbert space L belongs to $\overline{\mathrm{span}}^{w^*}(Y_0X_0)$.
Then $Y = \overline{\mathrm{span}}^{w^*}(Y_0B)$ is a right weak rigged module over B.*

Proof Clearly Y is a left $B-$module. We shall prove that it is a right $A-$module:

$$AY_0B \subset \overline{\mathrm{span}}^{\mathrm{w}^*}(Y_0X_0)AY_0B \subset \overline{\mathrm{span}}^{\mathrm{w}^*}(Y_0X_0AY_0B). \tag{2.1}$$

Since $X_0AY_0 \subset B$, (2.1) implies that

$$AY_0B \subset Y \Rightarrow AY \subset Y.$$

For every $M \in \mathcal{M}$, we write $\phi(M)$ for the projection onto $\overline{Y_0M(H)}$. Theorem 1.1 implies that $\phi(M) \in \mathcal{N}$ and $Y = Op(\phi)$.

Let X be the space

$$\{x \in B(L, H) : M^{\perp}x\phi(M) = 0 \;\; \forall M \;\; \in \;\; \mathcal{M}\},$$

and $\hat{X}$ be the space $\overline{\mathrm{span}}^{\mathrm{w}^*}(BX_0A)$. If $M \in \mathcal{M}$,

$$M^{\perp}BX_0AY_0M \subset M^{\perp}BM = 0 \Rightarrow M^{\perp}BX_0A\phi(M) = 0.$$

We proved that $\hat{X} \subset X$. On the other hand,

$$\overline{\mathrm{span}}^{\mathrm{w}^*}(Y\hat{X}) = \overline{\mathrm{span}}^{\mathrm{w}^*}(YBX_0A) \supset \overline{\mathrm{span}}^{\mathrm{w}^*}(Y_0X_0)A \supset A.$$

It follows that

$$\overline{\mathrm{span}}^{\mathrm{w}^*}(YX) \supset A \Rightarrow I_L \in \overline{\mathrm{span}}^{\mathrm{w}^*}(YX).$$

Theorems 2.1 and 2.2 imply that Y is a weak* rigged module over B. $\square$

Theorem 2.5. *Let $\mathcal{M}$ be a nest acting on the Hilbert space H and $B = \mathrm{Alg}(\mathcal{M})$ be the corresponding nest algebra. Suppose that there exist spaces $X_0 \subset B(L, H), Y_0 \subset B(H, L)$ such that:*
(i) $X_0Y_0 \subset B$;
(ii) The identity of the Hilbert space L belongs to $\overline{\mathrm{span}}^{w^*}(Y_0X_0)$.
It follows that the space $Y = \overline{\mathrm{span}}^{w^*}(Y_0B)$ is a right weak rigged module over B.*

Proof We define the map

$$\theta : \mathcal{M} \to pr(B(L)), \ \ \theta(M) = \overline{Y_0 M(H)}$$

and the algebra

$$\mathcal{A} = \mathrm{Alg}(\theta(\mathcal{M})) = \{a : \quad \theta(M)^{\perp} a \theta(M) = 0, \ \ \forall M \ \ \in \ \ \mathcal{M}\}.$$

Since $\theta(\mathcal{M})$ is a totally ordered set of projections, A is a nest algebra. By Lemma 2.4 it suffices to prove that $X_0 A Y_0 \subset B$.

For all $M \in \mathcal{M}, x \in X_0$ we have

$$x\theta(M)(H) \subset \overline{xY_0 M(H)} \subset \overline{BM(H)} \subset M(H).$$

So $M^{\perp} X_0 \theta(M) = 0 \ \forall \ M \ \ \in \ \ \mathcal{M}$. It follows that

$$M^{\perp} X_0 A Y_0 M = M^{\perp} X_0 A \theta(M) Y_0 M$$

$$= M^{\perp} X_0 \theta(M) A Y_0 M = 0, \ \ \forall \ M \ \ \in \ \ \mathcal{M} \Rightarrow X_0 A Y_0 \subset B$$

The proof is complete. $\square$

One consequence of our theory is the following proposition:

Proposition 2.6. *Let $\mathcal{M}$ be a nest acting on an infinite dimensional separable Hilbert space H. We assume that $\mathcal{M}$ is not totally atomic. Then there exists a right weak* rigged module Y over* $\mathrm{Alg}(\mathcal{M})$ *such that $M_l(Y) \cong \mathrm{Alg}(\mathcal{N})$, where $M_l(Y)$ is the algebra of left multipliers of Y and $\mathcal{N}$ is the nest of Volterra.*

Proof Let p be the supremum of the atoms of $\mathcal{M}$. Our assumptions imply that $p < I_H$. So the nest

$$\mathcal{M}|_{p(H)^{\perp}} = \{M|_{p(H)^{\perp}} : M \in \mathcal{M}\}$$

is a continuous nest acting on $p(H)^{\perp}$. Also, the map

$$\sigma : \mathcal{M} \to \mathcal{M}|_{p(H)^{\perp}} : M \to M|_{p(H)^{\perp}}$$

is a continuous nest homomorphism. All continuous nests are isomorphic with the nest of Volterra, [5]. Therefore there exists a nest isomorphism $\theta : \mathcal{M}|_{p(H)^{\perp}} \to \mathcal{N}$. The map $\tau = \theta \circ \sigma : \mathcal{M} \to \mathcal{N}$ is a continuous surjective nest homomorphism. Put $Y = Op(\tau)$. Theorem 2.1 implies that $M_l(Y) \cong \mathrm{Alg}(\mathcal{N})$. $\square$

3. Right weak* rigged modules over nest algebras

In this section we prove that a right weak* rigged module over a nest algebra is also a left nest algebra module, hence a nest algebra bimodule. We also prove that a right weak* rigged module over a nest algebra is $Op(\phi)$ for some continuous nest homomorphism ϕ. We need the following lemma:

Lemma 3.1. *Let $\mathcal{M}$ be a nest acting on the Hilbert space H, and let $B = \mathrm{Alg}(\mathcal{M})$ be the corresponding nest algebra and $Y \subset B(H, L)$ be a weak* closed space such that $YB \subset Y$. We also assume that there exists a space $X \subset B(L, H)$ such that $XY \subset B$ and $I_L \in \overline{\mathrm{span}}^{w^*}(YX)$. Then Y is a nest algebra bimodule, i.e., there exists a nest algebra $A \subset B(L)$ such that $AY \subset Y$.*

Proof Write $\theta : \mathcal{M} \to pr(B(L))$ for the map which sends $M \in \mathcal{M}$ to the projection onto $\overline{YM(H)}$, and denote by A the algebra $\mathrm{Alg}(\theta(\mathcal{M}))$, and put $\hat{Y} = Op(\theta)$. Clearly $Y \subset \hat{Y}$. We denote by $\hat{X}$ the space

$$\{x \in B(L, H) : \ \ M^{\perp} x \theta(M) = 0 \ \ \forall \ \ M \ \in \ \mathcal{M}\}.$$

If $x \in X$ and $y \in Y$, then since $xy \in B$ we have for all $M \in \mathcal{M}$ that

$$M^{\perp} xyM = 0 \Rightarrow M^{\perp} x\theta(M) = 0 \Rightarrow x \in \hat{X}.$$

We proved $X \subset \hat{X}$. Also for all $M \in \mathcal{M}$

$$M^{\perp} \hat{X}\hat{Y}M = M^{\perp} \hat{X}\theta(M)\hat{Y}M = 0 \Rightarrow \hat{X}\hat{Y} \subset B.$$

Since $A\hat{Y} \subset \hat{Y}$, it suffices to prove $\hat{Y} = Y$. Indeed,

$$\begin{aligned} \hat{Y} \ &\subset \overline{\mathrm{span}}^{\mathrm{w}^*}(YX)\hat{Y} \ \subset \ \overline{\mathrm{span}}^{\mathrm{w}^*}(YX\hat{Y}) \\ &\subset \ \overline{\mathrm{span}}^{\mathrm{w}^*}(Y\hat{X}\hat{Y}) \ \subset \ \overline{\mathrm{span}}^{\mathrm{w}^*}(YB) \ \subset \ Y. \qquad \square \end{aligned}$$

Theorem 3.2. *Let $\mathcal{M}$ be a nest acting on the Hilbert space H, let $B = \mathrm{Alg}(\mathcal{M})$ be the corresponding nest algebra, and let Y be a right weak* rigged module over B. Then there exists a Hilbert space K, a completely isometric weak* continuous $B-$module map $\Phi : Y \to B(H, K)$, and a nest algebra $A \subset B(K)$, such that $A\Phi(Y)B \subset \Phi(Y)$.*

Proof We recall the following facts from [1]. The space $K = Y \otimes_B^{\sigma h} H$ with its norm is a Hilbert space. The map $\Phi : Y \to B(H, K)$ given by

$$\Phi(y)(h) = y \otimes_B h$$

is a completely isometric, weak* continuous $B-$module map. We recall the space $\widetilde{X} = w^* CB(Y, B)_B$ from Section 2. The map

$$\Psi : \widetilde{X} \to B(K, H), \quad \Psi(u)(y \otimes_B h) = u(y)(h)$$

is also a complete isometry and weak* continuous. Finally

$$I_K \in \overline{\mathrm{span}}^{\mathrm{w}^*}(\Phi(Y)\Psi(\widetilde{X})).$$

We can easily check that

$$\Psi(\widetilde{X})\Phi(Y) \subset B$$

and

$$\Phi(Y)B \subset \Phi(Y).$$

So $\Phi(Y)$ satisfies the assumptions of Lemma 3.1. Therefore $\Phi(Y)$ is an $A-B$ bimodule for a nest algebra A acting on the Hilbert space K. $\square$

Theorem 3.3. *Let $\mathcal{M}$ be a nest acting on the Hilbert space H, let $B = \mathrm{Alg}(\mathcal{M})$ be the corresponding nest algebra, and let Y be a right dual operator $B-$module. Then the following are equivalent:*

(i) The space Y is a weak rigged module over B.*

(ii) There exists a Hilbert space K, a completely isometric weak continuous $B-$module map $\Phi : Y \to B(H, K)$, and a continuous nest homomorpism $\theta : \mathcal{M} \to pr(B(K))$, such that $\Phi(Y) = Op(\theta)$.*

Proof

(ii)$\Rightarrow$ (i):

By Theorem 2.1, $\Phi(Y)$ is a right weak* rigged module over B. Since $\Phi : Y \to \Phi(Y)$ is a completely isometric weak* continuous $B-$module map, Y is a right weak* rigged module over B.

(i)$\Rightarrow$ (ii):

Using Theorem 3.2, we may assume that $Y \subset B(H, L)$ for a Hilbert space L and there exists a nest $\mathcal{N}$ acting on L such that $AYB \subset Y$, where $A = \mathrm{Alg}(\mathcal{N})$. We recall the Hilbert space K, the space $\tilde{X}$ and the maps Φ, Ψ from the proof of Theorem 3.2.

There exist nets

$$(y_t)_t \subset R_\infty^{fin}(Y), \quad (u_t)_t \subset C_\infty^{fin}(\tilde{X})$$

such that $w^* - \lim_t \Phi(y_t)\Psi(u_t) = I_K$.

We define the map

$$\theta : \mathcal{M} \to pr(B(K)) : \theta(M) = \overline{\Phi(Y)M(H)}$$

which is left continuous, [11]. As in Theorem 2.2, we can prove that it is right continuous.

Let ϕ be the restriction of $\mathrm{Map}(Y)$ to $\mathcal{M}$. By Theorem 1.1, $\phi(\mathcal{M}) \subset \mathcal{N}$ and $Y = Op(\phi)$.

Put $\Omega = Op(\theta)$. It suffices to prove that $\Omega = \Phi(Y)$. Let T be the operator from K to L which sends $y \otimes_B h$ to $y(h)$. Observe that $T\Phi(y) = y$ for all $y \in Y$ and $T\theta(M)(K) \subset \phi(M)(H)$ for all $M \in \mathcal{M}$. If $S \in \Omega$ and $M \in \mathcal{M}$, then

$$TSM = T\theta(M)SM = \phi(M)T\theta(M)SM = \phi(M)TSM,$$

so $TS \in Y$. In the sequel, we can see that

$$\Phi(TS) = \lim_t \Phi(T\Phi(y_t)\Psi(u_t)S) = \lim_t \Phi(y_t\Psi(u_t)S).$$

Since $\Psi(u)S \in B$ for all $u \in \tilde{X}$ and Φ is a $B-$module map,

$$\Phi(TS) = \lim_t \Phi(y_t)\Psi(u_t)S = S.$$

We proved $\Omega \subset \Phi(Y)$. On the other hand

$$\Phi(y)M(h) = y \otimes_B M(h) \Rightarrow \theta(M)^\perp \Phi(y)M = 0, \ \ \forall M \in \mathcal{M}.$$

So $\Omega \supset \Phi(Y)$. $\square$

Remark 3.4. Let H, L be Hilbert spaces, $B \subset B(H)$ be a nest algebra, and $Y \subset B(H, L)$ be a right weak* rigged module over B. We also assume that K, Φ, θ are as in the above theorem. Then there exists, see the proof, a contraction $T : K \to L$ such that $T\Phi(y) = y$ for all $y \in Y$. Also we may consider that $\theta = \mathrm{Map}(\Phi(Y))$.

The following two propositions can be used to construct modules over nest algebras which are not weak* rigged modules.

Proposition 3.5. *Let $\mathcal{M}$ be a continuous nest acting on the Hilbert space H, B be the corresponding nest algebra, and Y be a right weak* rigged module over B. Then the algebra of left multipliers of Y, $M_l(Y)$, is isomorphic with an algebra* $\mathrm{Alg}(\mathcal{N})$ *where $\mathcal{N}$ is a continuous nest.*

Proof From Theorem 3.3, there exists a weak* continuous completely isometric $B-$module map $\Phi : Y \to B(H, K)$, a nest algebra $A = \mathrm{Alg}(\mathcal{N}) \subset B(K)$, and a continuous surjective nest homomorphism $\theta : \mathcal{M} \to \mathcal{N}$, such that $\Phi(Y) = Op(\theta)$. By Theorem 2.1, $A \cong M_l(Y)$. Clearly $\mathcal{N}$ is a continuous nest. $\square$

Proposition 3.6. *Let $\mathcal{M}$ (resp. $\mathcal{N}$) be a nest acting on the Hilbert space H (resp. K), let B, (resp. A) be the corresponding nest algebra, and let Y be a weak* closed $A - B$ bimodule, such that $\overline{Y(H)} = K$. We assume that $\mathcal{M}$ is a continuous nest and $\mathcal{N}$ has at least one atom. Then Y is not a right weak* rigged module over B.*

Proof Let q be an atom of $\mathcal{N}$. It follows that the space $qB(K)q$ is a subset of the algebra A. The space qY is a subspace of $B(H, q(K))$ and satisfies

$$B(q(K))qYB \subset qY.$$

So qY is a nest algebra bimodule over the algebras $B(q(K))$, corresponding to the nest $\{0_K, q\}$, and B. If

$$\theta = \mathrm{Map}(qY) : \mathcal{M} \to \{0_K, q\}$$

by Theorem 1.1

$$qY = \{y \in B(H, q(K)) : \theta(M)^\perp yM = 0, \ \ \forall\ M \ \ \in \ \ \mathcal{M}\}.$$

We define the projection

$$M_0 = \vee\{M \in \mathcal{M} : \ \ \theta(M) = 0\}.$$

Since θ is left continouous, $\theta(M_0) = 0$. If $M > M_0$, then $\theta(M) > 0$. Thus $\theta(M) = q$. Therefore

$$qY = \{y \in B(H, q(K)) : yM_0 = 0\} = B(H, q(K))M_0^{\perp}.$$

Assume now that Y is a weak* rigged module over B. Using for example Definition 1.2, we can verify that qY is a weak* rigged module over B. In this case,

$$M_l(qY) = B(q(K)) = \mathrm{Alg}(\{0_K, q\}).$$

This contradicts Proposition 3.5. $\square$

4. Examples

Example 4.1. *If $B = \mathrm{Alg}(\mathcal{M}) \subset B(H)$ is a nest algebra corresponding to a finite nest $\mathcal{M}$, $A = \mathrm{Alg}(\mathcal{N})$ is a nest algebra, and Y is a weak* closed $B - A$ bimodule, then Y is a right weak* rigged module over B. Indeed, the map $\phi : \mathcal{M} \to \mathcal{N}$ sending every $M \in \mathcal{M}$ to the projection onto $\overline{YM(H)}$ is continuous. Use now Theorems 1.1, 2.1.*

Example 4.2. *Let $B = \mathrm{Alg}(\mathcal{M}), A = \mathrm{Alg}(\mathcal{N})$ be nest algebras and $\phi : \mathcal{M} \to \mathcal{N}$ be a nest isomorphism. Since ϕ is continuous, the space $Y = Op(\phi)$ is a right weak* rigged module over B.*

Example 4.3. *Let H be an infinite dimensional separable Hilbert space and P_n be a strictly increasing sequence of projections such that $\vee_n P_n = I_H$. The set $\mathcal{M} = \{0_H, P_n, n \in \mathbb{N}, I_H\}$ is a nest. Suppose that $A = \mathrm{Alg}(\mathcal{N})$ is another nest algebra and Y is a weak* closed $A - \mathrm{Alg}(\mathcal{M})$ bimodule. Then Y is a right weak* rigged module over $\mathrm{Alg}(\mathcal{M})$. Indeed, the map*

$$\phi = \mathrm{Map}(Y) : \mathcal{M} \to \mathcal{N}$$

is continuous in every P_n and left continuous in I_H, by Theorem 1.1. Therefore ϕ is continuous.

Example 4.4. *Let B be a nest algebra and s be an invertible operator. By Theorem 2.5, $Y = sB$ is a right weak* rigged module over B.*

Example 4.5. *Let $\{e_n : n \in \mathbb{N}\}$ be an orthonormal basis of the Hilbert space H and p_n be the projection onto the space generated by the vectors $\{e_1, e_2, ..., e_n\}$ for all n. Let $\mathcal{M}$ be the nest $\{0_H, p_n, I_H, \quad n \in \mathbb{N}\}$, B be the algebra $\mathrm{Alg}(\mathcal{M})$, and x, y be the shift operators given by*

$$y(e_1) = 0, y(e_n) = e_{n-1}, \quad \forall n \geq 2,$$

$$x(e_n) = e_{n+1}, \quad \forall n \geq 1.$$

Since $x^m y^m$ is the projection onto $p_m(H)^{\perp}$ which belongs to B and $y^m x^m = I_H$ for all m, Theorem 2.5 then implies that $Y = \overline{y^m B}^{w^*}$ is a right weak rigged module over B for all m.*

Example 4.6. *Let B be a nest algebra and $\mathcal{T}$ be a ternary ring of operators such that $\mathcal{T}^*\mathcal{T} \subset B$. Since $\overline{\text{span}}^{w^*}(\mathcal{T}\mathcal{T}^*)$ is a unital algebra by Theorem 2.5 the space $\overline{\text{span}}^{w^*}(\mathcal{T}B)$ is a weak* rigged module over B. Modules of this type are called* projective weak* rigged modules *and they have special properties* [3].

The next example is an example of a nest algebra bimodule which is not a weak* rigged module.

Example 4.7. *Let $\mathcal{M}$ be a continuous nest acting on H, B be the algebra* $\text{Alg}(\mathcal{M})$*, and $Y = B(H, K)$ where K is another Hilbert space. The space Y is a nest algebra bimodule over $B(K)$ and B. But it is not a weak* rigged module over B because $\mathcal{M}$ is a continuous nest and the nest $\{0_K, I_K\}$ corresponding to $B(K)$ is totally atomic. (Use Proposition 3.6).*

This example is also an example of an $A - B$ bimodule Y where both A and B are continuous nest algebras and Y is not a weak* rigged module over B : Choose infinite dimensional Hilbert spaces K and H, and continuous nest algebras $B \subset B(K)$ and $A \subset B(L)$. In the same way we can prove that $Y = B(H, K)$ is not a weak* rigged module over B.

Example 4.8. *In Example 4.3, $\mathcal{M}$ is a totally atomic nest and Y is a right weak* rigged module over* $\text{Alg}(\mathcal{M})$*. As we can easily prove, $M_l(Y)$ is isomorphic to a nest algebra* $\text{Alg}(\mathcal{N})$ *where $\mathcal{N}$ is not a continuous nest. Here, we prove that there is a totally atomic nest $\mathcal{M}$ and a right weak* rigged module Y over $B =$* $\text{Alg}(\mathcal{M})$ *such that $M_l(Y) \cong$* $\text{Alg}(\mathcal{N}_v)$ *where $\mathcal{N}_v$ is the nest of Volterra:*

There is a totally atomic nest $\mathcal{M}$ (the nest of Cantor) and an isomorphic nest $\mathcal{M}_2$ which is not totally atomic, [5, Example 7.19]. *Suppose that $\rho : \mathcal{M} \to \mathcal{M}_2$ is the above nest isomorphism. Let $\tau : \mathcal{M}_2 \to \mathcal{N}_v$ be the surjective continuous nest homomorphism described in Proposition 2.6. If $Y = Op(\tau \circ \rho)$, Theorem 2.1 implies that $M_l(Y) \cong$* $\text{Alg}(\mathcal{N}_v)$.

5. Spatially embedding algebras

In this section we investigate a weaker definition than Definition 1.2. The objects of the theory are the reflexive algebras and more specifically the CSL algebras. A nest algebra is a special type of a CSL algebra. See [5] or in [17] for the definition of reflexivity and the notions of CSL, the CSL algebra and a reflexive algebra.

Definition 5.1. *Let A (resp. B) be a weak* closed algebra acting on H (resp. K). We say that A is* ***spatially embedded*** *in B if there exist a $B - A$ bimodule $X \subset B(H, K)$ and an $A - B$ bimodule $Y \subset B(K, H)$ such that*

(i) $XY \subset B$,
(ii) $A = \overline{\text{span}}^{w^*}(YX)$.
Moreover, if $B = \overline{\text{span}}^{w^*}(XY)$, *we call* A *and* B ***spatially Morita equivalent***.

Remark 5.1. *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be nests corresponding to the algebras* B *and* A *and let* $\phi : \mathcal{M} \to \mathcal{N}$ *be a continuous surjective nest homomorphism. It follows that* A *is embedded spatially in* B *(see Theorem 2.2). On the other hand, if* $B = \mathrm{Alg}(\mathcal{M})$ *is a nest algebra and* Y *is a right weak* rigged module over* B, *there is a normal completely isometric representation* Φ *of* Y *and a nest algebra* $A = \mathrm{Alg}(\mathcal{N})$ *such that* $\Phi(Y)$ *is an* $A - B$ *bimodule and* A *is spatially embedded in* B *(see Theorems 3.2 and 3.3). In this case there exists a continuous nest homomorphism from* $\mathcal{M}$ *onto* $\mathcal{N}$.

Proposition 5.2. *Let* A, B, X, Y *be as in Definition 5.1. Put*

$$\Omega_2 = M_2(B), \quad \Omega_1 = \begin{pmatrix} B & X \\ Y & A \end{pmatrix}.$$

The algebras Ω_1 *and* Ω_2 *are spatially Morita equivalent.*

Proof We define spaces

$$\hat{Y} = \begin{pmatrix} B & B \\ Y & Y \end{pmatrix}, \quad \hat{X} = \begin{pmatrix} B & X \\ B & X \end{pmatrix}.$$

We can easily check that $\hat{Y}$ is an $\Omega_1 - \Omega_2$ bimodule, $\hat{X}$ is an $\Omega_2 - \Omega_1$ bimodule, and

$$\Omega_2 = \overline{\text{span}}^{\text{w}^*}(\hat{X}\hat{Y}), \quad \Omega_1 = \overline{\text{span}}^{\text{w}^*}(\hat{Y}\hat{X}). \qquad \square$$

Corollary 5.3. *Let* A, B, X, Y *be as in Definition 5.1. If* B *is a reflexive algebra, then* X, Y, *and* A *are reflexive spaces.*

Proof If B is reflexive, $M_2(B)$ is reflexive. It follows from [7, remark 4.2] that the algebra Ω_1 defined in Proposition 5.2 is reflexive. Therefore X, Y, and A are reflexive. $\square$

In the rest of this section, if Z is a set of operators, then $\mathcal{K}(Z)$ denotes its subset of compact operators, $\mathcal{F}(Z)$ denotes its subset of finite rank operators, and $\mathcal{R}(Z)$ denotes its subset of rank one operators.

Theorem 5.4. *Let* A, B, X, Y *be as in Definition 5.1.*
(i) If $B = \overline{\mathcal{K}(B)}^{w^*}$, *then* $A = \overline{\mathcal{K}(A)}^{w^*}$.
(ii) If $B = \overline{\mathcal{F}(B)}^{w^*}$, *then* $A = \overline{\mathcal{F}(A)}^{w^*}$.
(iii) If $B = \overline{\text{span}}^{w^*}(\mathcal{R}(B))$, *then* $A = \overline{\text{span}}^{w^*}(\mathcal{R}(A))$.

Proof Suppose that $B = \overline{\mathcal{K}(B)}^{w^*}$. Since Y is a left B-module,

$$Y = \overline{\mathrm{span}}^{\mathrm{w}^*}(YB) = \overline{\mathrm{span}}^{\mathrm{w}^*}(Y\mathcal{K}(B)) = \overline{\mathcal{K}(Y)}^{w^*}.$$

Therefore

$$A = \overline{\mathrm{span}}^{\mathrm{w}^*}(YX) = \overline{\mathrm{span}}^{\mathrm{w}^*}(\mathcal{K}(Y)X) = \overline{\mathcal{K}(A)}^{w^*}.$$

The proofs of (ii) and (iii) are similar. $\square$

Theorem 5.5. *Let A, B, X, Y be as in Definition 5.1. Assume that A is a unital algebra.*

(i) If $s \in \mathcal{K}(A)$ is a nonzero operator, then B contains a nonzero compact operator.

(ii) If $s \in \mathcal{F}(A)$ is a nonzero operator, then B contains a nonzero finite rank operator.

(iii) If $s \in \mathcal{R}(A)$, then B contains a rank one operator.

Proof Let $s \in \mathcal{K}(A)$ be a nonzero operator. There exist $x_0 \in X, y_0 \in Y$ such that $x_0 s y_0 \neq 0$. Indeed, if $XsY = 0$ then $YXsYX = 0$. By assumption, $\overline{\mathrm{span}}^{\mathrm{w}^*}(YX)$ contains the identity operator. So $s = 0$. This is a contradiction.

Observe that $0 \neq x_0 s y_0 \in \mathcal{K}(B)$, so statement (i) holds. The proofs of statements (ii), (iii) are similar. $\square$

Theorem 5.6. *Let $\mathcal{L}_1, \mathcal{L}_2$ be CSLs acting on the Hilbert spaces K, H respectively and A, B be the corresponding CSL algebras. The following are equivalent:*

(i) A is embedded spatially in B.

(ii) There exists a continuous surjective lattice homomorphism $\phi : \mathcal{L}_2 \to \mathcal{L}_1$.

Proof

(i)$\Rightarrow$(ii):

Suppose that there exist spaces X and Y satisfying (i) and (ii) of the definition 5.1. We define the spaces $\hat{X}, \hat{Y}$ and the algebras Ω_1, Ω_2 as in Proposition 5.2. In the sequel if C is a unital algebra, we denote by $\mathrm{Lat}(C)$ the lattice of invariant projections of C. If $\sigma = \mathrm{Map}(\hat{Y})$, Theorem 4.1 in [7] implies that σ is a lattice isomorphism from $\mathrm{Lat}(\Omega_2)$ onto $\mathrm{Lat}(\Omega_1)$.

If $P \in \mathcal{L}_2$ we denote by $\theta(P)$ the projection onto $\overline{YP(H)}$, which clearly belongs to $\mathcal{L}_1$. We can easily check that

$$\sigma((P \oplus P)) = P \oplus \theta(P)$$

for all $P \in \mathcal{L}_2$. So the map $\theta : \mathcal{L}_2 \to \mathcal{L}_1$ is a continuous CSL homomorphism. It remains to show that θ is surjective. If $Q \in \mathcal{L}_1$, then $\overline{AQ(K)} = Q(K)$. The

projection onto the closure of the linear span of the set

$$\{\Omega_1 \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} : \ \xi \in H, \quad \eta \in K\}$$

belongs to

$$\mathrm{Lat}(\Omega_1) = \{(P \oplus \theta(P)) : \ P \ \in \ \mathcal{L}_2\}.$$

So it is equal to a projection $R = (P_0 \oplus \theta(P_0))$ for $P_0 \in \mathcal{L}_2$.

Observe that R is the projection onto the space generated by the vectors of the form

$$\begin{pmatrix} 0 & xQ \\ 0 & aQ \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} : \ \xi \in H, \quad \eta \in K, x \in X, \ a \in A.$$

So $\theta(P_0)$ is the projection onto $\overline{AQ(K)}$ which is Q. Therefore θ is surjective.

(ii)$\Rightarrow$ (i):

Let $\phi : \mathcal{L}_2 \to \mathcal{L}_1$ be a continuous surjective lattice homomorphism. We define the CSLs

$$\mathcal{N}_2 = \{P \oplus P : \ P \ \in \ \mathcal{L}_2\}, \quad \mathcal{N}_1 = \{P \oplus \phi(P) : \ P \ \in \ \mathcal{L}_2\}.$$

The map $\rho : \mathcal{N}_2 \to \mathcal{N}_1$ sending every $P \oplus P$ to $P \oplus \phi(P)$ is a CSL isomorphism. We define the spaces

$$\begin{aligned}
\hat{Y} =& \{y \in B(K \oplus K, K \oplus H) : \rho(L)^{\perp} y L = 0, \ \ \forall L \in \mathcal{N}_2\}, \\
\hat{X} =& \{x \in B(K \oplus H, K \oplus K) : L^{\perp} x \rho(L) = 0, \ \ \forall L \in \mathcal{N}_2\}, \\
Y =& \{y \in B(K, H) : \phi(P)^{\perp} y P = 0, \ \ \forall P \in \mathcal{L}_2\}, \\
X =& \{x \in B(H, K) : P^{\perp} x \phi(P) = 0, \ \ \forall P \in \mathcal{L}_2\}.
\end{aligned}$$

We can easily verify that

$$\hat{Y} = \begin{pmatrix} B & B \\ Y & Y \end{pmatrix}, \quad \hat{X} = \begin{pmatrix} B & X \\ B & X \end{pmatrix}, \quad \mathrm{Alg}(\mathcal{N}_1) = \begin{pmatrix} B & X \\ Y & A \end{pmatrix}.$$

By Theorem 4.3 in [7], we conclude that

$$\mathrm{Alg}(\mathcal{N}_1) = \overline{\mathrm{span}}^{\mathrm{w}^*}(\hat{Y}\hat{X}),$$

therefore $A = \overline{\mathrm{span}}^{\mathrm{w}^*}(YX)$. The proof is complete. $\square$

There is no known lattice condition corresponding to the weak* density of the compact operators in a CSL algebra. But comparing Theorems 5.4 and 5.6, we have the following corollary:

Corollary 5.7. *Let $\mathcal{L}_1, \mathcal{L}_2$ be CSLs and $\phi : \mathcal{L}_2 \to \mathcal{L}_1$ be a continuous surjective lattice homomorphism.*

(i)If $\mathrm{Alg}(\mathcal{L}_2) = \overline{\mathcal{K}(\mathrm{Alg}(\mathcal{L}_2))}^{w^*}$, *then* $\mathrm{Alg}(\mathcal{L}_1) = \overline{\mathcal{K}(\mathrm{Alg}(\mathcal{L}_1))}^{w^*}$.

(ii)If $\mathrm{Alg}(\mathcal{L}_2) = \overline{\mathcal{F}(\mathrm{Alg}(\mathcal{L}_2))}^{w^*}$, *then* $\mathrm{Alg}(\mathcal{L}_1) = \overline{\mathcal{F}(\mathrm{Alg}(\mathcal{L}_1))}^{w^*}$.

Similarly Theorems 5.5 and 5.6 imply

Corollary 5.8. *Let $\mathcal{L}_1, \mathcal{L}_2$ be CSLs and $\phi : \mathcal{L}_2 \to \mathcal{L}_1$ be a continuous surjective lattice homomorphism. If $s \in \mathrm{Alg}(\mathcal{L}_1)$ is, respetively, a nonzero compact operator, a nonzero finite rank operator, a rank one operator, then $\mathrm{Alg}(\mathcal{L}_2)$ contains, respectively, a nonzero compact operator, a nonzero finite rank operator, a rank one operator.*

If U is a reflexive and separably acting bimodule over maximal abelian selfadjoint algebras (*masa*)s, there exists a smallest weak* closed masa bimodule U_{min} whose reflexive hull is U. In the special case $U = U_{min}$, we call U *synthetic*. A CSL $\mathcal{L}$ is called *synthetic* if the algebra $\mathrm{Alg}(\mathcal{L})$ is synthetic [5]. We present the following relevant result:

Theorem 5.9. *Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be separably acting CSLs, corresponding to the algebras B and A, and suppose $\phi : \mathcal{L}_2 \to \mathcal{L}_1$ is a continuous surjective lattice homomorphism. If $\mathcal{L}_2$ is synthetic, then $\mathcal{L}_1$ is synthetic.*

Proof We define the CSLs

$$\mathcal{N}_2 = \{P \oplus P : \ P \in \ \mathcal{L}_2\}, \quad \mathcal{N}_1 = \{P \oplus \phi(P) : \ P \in \ \mathcal{L}_2\}.$$

The map

$$\rho : \mathcal{N}_2 \to \mathcal{N}_1, \quad P \oplus P \to P \oplus \phi(P)$$

is a CSL isomorphism. We have that

$$\mathrm{Alg}(\mathcal{N}_1) = \begin{pmatrix} B & Y \\ X & A \end{pmatrix}, \tag{5.1}$$

where X and Y are defined as in Theorem 5.6. As in the proof of Theorem 4.7 in [7], we can prove that

$$\mathrm{Alg}(\mathcal{N}_1)_{min} = \begin{pmatrix} B_{min} & Y_{min} \\ X_{min} & A_{min} \end{pmatrix}. \tag{5.2}$$

If $\mathcal{L}_2$ is synthetic then $\mathcal{N}_2$ is synthetic. By Theorem 4.7 in [7], $\mathcal{N}_1$ is synthetic, so by (5.1) and (5.2), $A = A_{min}$. $\square$

References

[1] D. P. Blecher and U. Kashyap, Morita equivalence of dual operator algebras, *J. Pure Appl. Algebra*, 212 (2008), 2401–2412.

[2] D. P. Blecher and U. Kashyap, A characterization and a generalization of W^*-modules, *Trans. Amer. Math. Soc.* , 363 (2011), 345-363.

[3] D. P. Blecher and J. E. Kraus, On a generalization of W^*-modules, *Banach Center Publ*, 91 (2010), 77–86.

[4] D. P. Blecher and C. Le Merdy, *Operator Algebras and Their Modules—An Operator Space Approach*, Oxford University Press, 2004.

[5] K. R. Davidson, *Nest Algebras*, Longman Scientific & Technical, Harlow, 1988.

[6] E. Effros and Z.-J. Ruan, *Operator Spaces*, Clarendon Press, Oxford, 2000.

[7] G. K. Eleftherakis, TRO equivalent algebras, *Houston J. of Mathematics*, to appear, Arxiv: math/0607488

[8] G. K. Eleftherakis, A Morita type equivalence for dual operator algebras, *J. Pure Appl. Algebra* 212 (2008) no. 5, 1060–1071.

[9] G. K. Eleftherakis, Morita equivalence of nest algebras, *Math. Scand.*, to appear, Arxiv: 1002.2335.

[10] G. K. Eleftherakis and V. I. Paulsen, Stably isomorphic dual operator algebras, *Math. Ann.* 341 (2008), no. 1, 99–112.

[11] J. A. Erdos amd S. C. Power, Weakly closed ideals of nest algebras, *J. Operator Theory* 7 (1982), 219–235.

[12] I. Kaplansky, Modules over operator algebras, *Amer. J. Math.* 75 (1953), 839–858.

[13] U. Kashyap, A Morita theorem for dual operator algebras, *J. Funct. Analysis*, 256 (2009), 3545–3567.

[14] W. L. Paschke, Inner product modules arising from compact groups of a von Neumann algebra, *Trans. Amer. Math. Soc.* 224 (1976), 87–102.

[15] V. I. Paulsen, *Completely Bounded Maps and Operator Algebras*, Cambridge University Press, 2002.

[16] G. Pisier, *Introduction to Operator Space Theory*, Cambridge University Press, 2003.

[17] V. S. Shulman and L. Turowska, Operator synthesis I. Synthetic sets, bilattices and tensor algebras, *J. Funct. Anal.* 209 (2004), 293-331.

Department of Mathematics, University of Athens, Panepistimioupolis 157 84, Athens, Greece.

E-mail address: `gelefth@math.uoa.gr`